

# AN ALGORITHM FOR THE NORMAL BUNDLE OF RATIONAL MONOMIAL CURVES

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**ABSTRACT.** In this short note we give an algorithm for calculating the splitting type of the normal bundle of any rational monomial curve. The algorithm is achieved by reducing the calculus to a combinatorial problem and then by solving it.

## 1. INTRODUCTION

It is well known that, up to projective transformations, any degree  $d$  rational curve  $C$  in  $\mathbb{P}^s(\mathbb{C})$  ( $d > s \geq 3$ ), which we will assume smooth or having at most ordinary singularities, is a suitable projection of the rational normal curve  $\Gamma_d$  of degree  $d$  in  $\mathbb{P}^d(\mathbb{C})$  from a projective linear space  $L$  of dimension  $d - s - 1$ . Moreover the normal bundle  $\mathcal{N}_C$  splits as a direct sum of line bundles  $\mathcal{O}_{\mathbb{P}^2}(\xi_1) \oplus \mathcal{O}_{\mathbb{P}^2}(\xi_2) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^2}(\xi_{s-1})$  where  $\xi_i$  are suitable integers. In principle, one should calculate these integers for any chosen  $L$ .

In [A-R2] the authors develop a general method to do this calculation. This method was previously used in [A-R1] to get the splitting type of the restricted tangent bundle of  $C$ . However, while for the tangent bundle it is possible to get an easy formula (see Theorem 3 of [A-R1]), for the normal bundle it is not possible to obtain the integers  $\xi_i$  without performing an often cumbersome calculation of the dimension of the kernel of a linear map (see Theorem 1 of [A-R2]). Of course, for a fixed curve, one can do the calculation with the help of a computer and moreover the method allows to prove some results about the Hilbert schemes of rational curves having a given splitting type (see §6 of [A-R2]), but a direct formula would be very useful.

We believe that, in general, it is very hard to accomplish this task with the above method, however it is possible to solve the problem for a special type of rational curves: the monomial ones, i. e. when the morphism  $f : \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^s(\mathbb{C})$  is given by monomials of the same degree in two variables. This type of rational curves is very investigated in the literature. In this paper we will give a formula for calculating the integers  $\xi_i$  when  $C$  is a monomial curve (see Theorem 5.4).

In §2 we fix notations and we recall some results from [A-R1] and [A-R2]. In §3 we address the case of monomial curves. In §4 we prove that the calculation reduces to a combinatorial problem. In §5 we will give a formula to solve the problem.

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## 2. NOTATION AND BACKGROUND MATERIAL

For us, a rational curve  $C \subset \mathbb{P}^s(\mathbb{C})$  will be the target of a morphism  $f : \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^s(\mathbb{C})$ . We will work always over  $\mathbb{C}$ . We will always assume that  $C$  is not contained in any hyperplane and that it has at most ordinary singularities. Let us put  $d := \deg(C) > s \geq 3$ . Let  $\mathcal{I}_C$  be the ideal sheaf of  $C$ , then  $\mathcal{N}_C := \text{Hom}_{\mathcal{O}_C}(\mathcal{I}_C/\mathcal{I}_C^2, \mathcal{O}_C)$  as usual and, taking the differential of  $f$ , we get:

$$0 \rightarrow \mathcal{T}_{\mathbb{P}^1} \rightarrow f^*\mathcal{T}_{\mathbb{P}^s} \rightarrow f^*\mathcal{N}_C \rightarrow 0$$

where  $\mathcal{T}$  denotes the tangent bundle. Of course we can always write:

$$\begin{aligned} \mathcal{T}_f &:= f^*\mathcal{T}_{\mathbb{P}^s} = \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(b_i + d + 2) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus(s-r)}(d + 1) \\ \mathcal{N}_f &:= f^*\mathcal{N}_C = \bigoplus_{i=1}^{s-1} \mathcal{O}_{\mathbb{P}^1}(c_i + d + 2) \end{aligned}$$

for suitable integers  $b_i \geq 0$  (see (14) of [A-R1]) and  $c_i \geq 0$  (see Proposition 10 of [A-R2] where we assumed  $c_1 \geq \dots \geq c_{s-1}$ ).

Every curve  $C$  is, up to a projective transformation, the projection in  $\mathbb{P}^s$  of a  $d$ -Veronese embedding  $\Gamma_d$  of  $\mathbb{P}^1$  in  $\mathbb{P}^d := \mathbb{P}(V)$  from a  $(d-s-1)$ -dimensional projective space  $L := \mathbb{P}(T)$  where  $V$  and  $T$  are vector spaces of dimension, respectively,  $d+1$  and  $e+1 := d-s$ . For any vector  $0 \neq v \in V$  let  $[v]$  be the corresponding point in  $\mathbb{P}(V)$ . Of course we require that  $L \cap \Gamma_d = \emptyset$  as we want that  $f$  is a morphism.

Let us denote by  $U = \langle x, y \rangle$  a fixed 2-dimensional vector space such that  $\mathbb{P}^1 = \mathbb{P}(U)$ , then we can identify  $V$  with  $S^d U$  ( $d$ -th symmetric power) in such a way that the rational normal degree  $d$  curve  $\Gamma_d$  can be considered as the set of pure tensors of degree  $d$  in  $\mathbb{P}(S^d U)$  and the  $d$ -Veronese embedding is the map

$$\alpha x + \beta y \rightarrow (\alpha x + \beta y)^d \quad (\alpha : \beta) \in \mathbb{P}^1.$$

From now on, any degree  $d$  rational curve  $C$ , will be determined (up to projective equivalences which are not important in our context) by the choice of a proper subspace  $T \subset S^d U$  such that  $\mathbb{P}(T) \cap \Gamma_d = \emptyset$ .

By arguing in this way, the elements of a base of  $T$  can be thought as homogeneous, degree  $d$ , polynomials in  $x, y$ . In [A-R1] and [A-R2] the authors relate the polynomials of any base of  $T$  with the splitting type of  $\mathcal{T}_f$  and  $\mathcal{N}_f$ . To describe this relation we need some additional definitions.

Let us indicate by  $\langle \partial_x, \partial_y \rangle$  the dual space  $U^*$  of  $U$ , where  $\partial_x$  and  $\partial_y$  indicate the partial derivatives with respect to  $x$  and  $y$ .

**Definition 2.1.** *Let  $T$  be any proper subspace of  $S^d U$ . Then:*

$$\begin{aligned} \partial T &:= \langle \omega(T) | \omega \in U^* \rangle \\ \partial^{-1} T &:= \bigcap_{\omega \in U^*} \omega^{-1} T \\ r(T) &:= \dim(\partial T) - \dim(T). \end{aligned}$$

Note that Definition 2.1 allows to define also  $\partial^k T$  and  $\partial^{-k} T$  for any integer  $k \geq 1$ , by induction. Moreover we can set  $\partial^0 T := T$ . Let us recall the following:

**Theorem 2.1.** *Let  $T \subset S^d U$  be any proper subspace as above such that  $\mathbb{P}(T) \cap \Gamma_d = \emptyset$ . Then  $r(T) \geq 1$  and there exist  $r$  polynomials  $p_1, \dots, p_r$  of degree  $d + b_1, \dots, d + b_r$  respectively, with  $b_i \geq 0$  and  $[p_i] \in \mathbb{P}^{d+b_i} \setminus \text{Sec}^{b_i}(\Gamma_{d+b_i})$  for  $i = 1, \dots, r$ , such that:*

$$\begin{aligned} T &= \partial^{b_1}(p_1) \oplus \partial^{b_2}(p_2) \oplus \dots \oplus \partial^{b_r}(p_r) \text{ and} \\ \partial T &= \partial^{b_1+1}(p_1) \oplus \partial^{b_2+1}(p_2) \oplus \dots \oplus \partial^{b_r+1}(p_r). \end{aligned}$$

*Proof.* It follows from Theorem 1 of [A-R1], because from our assumptions  $S_T = 0$  in the notation of [A-R1]. Recall that  $\text{Sec}^b(\Gamma_{d+b})$  is the variety generated by sets of  $b + 1$  distinct points of  $\Gamma_{d+b}$ .  $\square$

From the above decomposition of  $T$  it is possible to get directly the splitting type of  $\mathcal{T}_f$  depending by the integers  $b_i$ , (see Theorem 3 of [A-R1]) however here we are interested in the splitting type of  $\mathcal{N}_f$ . To this aim the following Proposition is useful:

**Proposition 2.2.** *In the above notations, for any integer  $k \geq 0$ , let us call  $\varphi(k) := h^0(\mathbb{P}^1, \mathcal{N}_f(-d-2-k))$ . Then the splitting type of  $\mathcal{N}_f$  is completely determined by  $\Delta^2[\varphi(k)] := \varphi(k+2) - 2\varphi(k+1) + \varphi(k)$ .*

*Proof.* We know that  $\mathcal{N}_f(-d-2) = \bigoplus_{i=1}^{s-1} \mathcal{O}_{\mathbb{P}^1}(c_i)$ , so that we have only to determine the integers  $c_i$ . By definition,  $\Delta^2[\varphi(k)]$  is exactly the number of integers  $c_i$  which are equal to  $k$ .  $\square$

From Proposition 2.2 it follows that to know the splitting type of  $\mathcal{N}_f$  it suffices to know  $\varphi(k)$  for any  $k \geq 0$ .

Let us consider the linear operators  $D_k : S^k U \otimes S^d U \rightarrow S^{k-1} U \otimes S^{d-1} U$ , such that  $D_k := \partial_x \otimes \partial_y - \partial_y \otimes \partial_x$ , and  $D_k^2 : S^k U \otimes S^d U \rightarrow S^{k-2} U \otimes S^{d-2} U$ . Of course, as  $T \subset S^d U$ , we can restrict  $D_k^2$  to  $S^k U \otimes T$  and we get a linear map  $D_{k|S^k U \otimes T}^2 : S^k U \otimes T \rightarrow S^{k-2} U \otimes \partial^2 T$ ; let us define:

$$T_k := \ker(D_{k|S^k U \otimes T}^2).$$

Then we have the following:

**Theorem 2.3.** *In the above notations:*

$$\varphi(0) = d + e$$

$$\varphi(1) = 2(e + 1)$$

$$\varphi(2) = 3(e + 1) - \dim(\partial^2 T)$$

$$\text{and for any } k \geq 2, \varphi(k) = \dim(T_k).$$

*Moreover the number of integers  $c_i$  such that  $c_i = 0$  is  $d - 1 - \dim(\partial^2 T)$ .*

*Proof.* See Theorem 1 and Proposition 11 of [A-R2]; note that, for  $k = 2$ , there are two different ways to get  $\varphi(2)$ .

By Proposition 2.2 the number of integers  $c_i$  such that  $c_i = 0$  is  $\Delta^2[\varphi(0)] = d - 1 - \dim(\partial^2 T)$ .  $\square$

To calculate  $\varphi(k)$  for  $k \geq 2$ , we can use the following very useful:

**Proposition 2.4.** *Let us assume that there is a direct decomposition  $T = T' \oplus T''$  such that  $\partial^2 T = \partial^2 T' \oplus \partial^2 T''$  and let us define*

$$K' := \ker(D_k^2 : S^k U \otimes T' \rightarrow S^{k-2} U \otimes \partial^2 T')$$

$$K'' := \ker(D_k^2 : S^k U \otimes T'' \rightarrow S^{k-2} U \otimes \partial^2 T'').$$

*Then, for any  $k \geq 2$ , we have  $\varphi(k) = \dim(K') + \dim(K'')$ .*

*Proof.* See Lemma 13 of [A-R2].  $\square$

Proposition 2.4 allows to simplify the calculation of  $\varphi(k)$ , however we need a method to evaluate  $\dim(K')$  and  $\dim(K'')$  for any possible above decomposition. To this aim, we introduce the following polarization (linear) maps  $p_k : S^{d+k} U \rightarrow S^k U \otimes S^d U$  such that, for any polynomial  $f \in S^{d+k} U$

$$p_k(f) = \frac{(\deg f - k)!}{\deg f!} \sum_{i=0}^k \binom{k}{i} x^{k-i} y^i \otimes \partial_x^{k-i} \partial_y^i(f).$$

We have the following:

**Proposition 2.5.** *In the above notations, for any integer  $k \geq 2$ , we have the following exact sequences of vector spaces:*

i)  $0 \rightarrow S^{d+k}U \rightarrow S^kU \otimes S^dU \rightarrow S^{k-1}U \otimes S^{d-1}U \rightarrow 0$  where the first map is  $p_k$  and the second map is  $D_k$ ;

ii)  $0 \rightarrow p_k(\partial^{-k}T) \rightarrow T_k \rightarrow p_{k-1}(N_k) \rightarrow 0$  where  $N_k$  is a suitable subspace of  $S^{d+k-2}U$ , the first map is an inclusion and the second map is the restriction of  $D_k$  to  $T_k$ ;

iii)  $0 \rightarrow p_{k-1}(\partial^{-k+1}\partial T) \rightarrow S^{k-1}U \otimes \partial T \rightarrow D_{k-1}(S^{k-1}U \otimes \partial T) \rightarrow 0$  where the first map is an inclusion and the second one is  $D_{k-1}$ ;

iv)  $0 \rightarrow p_{k-1}(N_k) \rightarrow p_{k-1}(\partial^{-k+1}\partial T) \rightarrow Q_k := \frac{p_{k-1}(\partial^{-k+1}\partial T)}{p_{k-1}(N_k)} \rightarrow 0$  where the first map is an inclusion.

Moreover we have the following commutative diagram where the left horizontal maps are inclusions and the lower vertical maps are induced by  $D_k$  and  $D_{k-1}$ .

$$\begin{array}{ccccccc}
& 0 & & 0 & & 0 & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 & \rightarrow & p_{k-1}(N_k) & \rightarrow & p_{k-1}(\partial^{-k+1}\partial T) & \rightarrow & Q_k \rightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & D_k(S^kU \otimes T) & \rightarrow & S^{k-1}U \otimes \partial T & \rightarrow & \frac{S^{k-1}U \otimes \partial T}{D_k(S^kU \otimes T)} \rightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & D_k^2(S^kU \otimes T) & \rightarrow & D_{k-1}(S^{k-1}U \otimes \partial T) & \rightarrow & \frac{D_{k-1}(S^{k-1}U \otimes \partial T)}{D_k^2(S^kU \otimes T)} \rightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

*Proof.* i) For any  $k \geq 1$  the maps  $D_k$  are surjective (see [A-R2] Corollary 6). It is easy to see that every  $p_k$  is injective and that  $p_k(S^{d+k}U) \subseteq \ker D_k$ . Then, by calculating the dimensions of these vector spaces we have  $p_k(S^{d+k}U) = \ker D_k$ .

ii) Let us consider  $D_{k|T_k} : T_k \rightarrow S^{k-1}U \otimes \partial T \subseteq S^{k-1}U \otimes S^{d-1}U$ ; by definition,  $D_{k|T_k}(T_k) \subseteq (S^{k-1}U \otimes \partial T) \cap \ker D_k = (S^{k-1}U \otimes \partial T) \cap p_{k-1}(S^{d+k-2}U)$ . As  $p_{k-1}$  is injective there exists a suitable subspace  $N_k \subseteq S^{d+k-2}U$  such that  $D_{k|T_k}(T_k) = p_{k-1}(N_k)$ . Moreover  $\ker D_{k|T_k} = \ker D_k \cap T_k = p_k(S^{d+k}U) \cap T_k = p_k\{f \in S^{d+k}U \mid \partial_x^{k-i} \partial_y^i(f) \in T \text{ for } i = 0, \dots, k\} = p_k(\partial^{-k}T)$ .

iii) Of course  $S^{k-1}U \otimes \partial T \rightarrow D_{k-1}(S^{k-1}U \otimes \partial T)$  is surjective; the kernel of this map is  $(S^{k-1}U \otimes \partial T) \cap \ker D_{k-1} = (S^{k-1}U \otimes \partial T) \cap p_{k-1}(S^{d+k-2}U) = p_{k-1}\{f \in S^{d+k-2}U \mid \partial_x^{k-1-i} \partial_y^i(f) \in \partial T \text{ for } i = 0, \dots, k-1\} = p_{k-1}(\partial^{-k+1}\partial T)$ .

iv) We have only to prove that  $p_{k-1}(N_k) \subseteq p_{k-1}(\partial^{-k+1}\partial T)$  i.e.  $D_{k|T_k}(T_k) \subseteq (S^{k-1}U \otimes \partial T) \cap \ker D_{k-1}$  and it follows from the definition of  $T_k$ .

From the above sequences we get the following diagram:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \rightarrow & p_k(\partial^{-k}T) & \rightarrow & p_k(\partial^{-k}T) & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & T_k & \rightarrow & S^k U \otimes T & \rightarrow & D_k^2(S^k U \otimes T) \rightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & p_{k-1}(N_k) & \rightarrow & D_k(S^k U \otimes T) & \rightarrow & D_k^2(S^k U \otimes T) \rightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

where the left vertical sequence is *ii*), the central vertical sequence follows immediately by *iii*), the central horizontal sequence follows from the definition of  $T_k$  and the lower horizontal sequence follows from the snake lemma.

By the above diagram, by recalling that there are obvious inclusions  $D_k(S^k U \otimes T) \rightarrow S^{k-1} U \otimes \partial T$  and  $D_k^2(S^k U \otimes T) \rightarrow D_{k-1}(S^{k-1} U \otimes \partial T)$  and by using the snake lemma we get the diagram of Proposition 2.5.  $\square$

**Corollary 2.6.** *By Proposition 2.5 we get that, for any  $k \geq 2$ ,  $\varphi(k) = \dim \partial^{-k} T + \dim \partial^{-k+1} \partial T - \dim(Q_k)$  and that  $Q_k \simeq \frac{p_{k-1}(\partial^{-k+1} \partial T)}{D_k(S^k U \otimes T) \cap p_{k-1}(\partial^{-k+1} \partial T)}$ .*

*Proof.* The evaluation for  $\varphi(k)$  follows from sequences *ii*) and *iv*) of Proposition 2.5. The isomorphism for  $Q_k$  follows from the diagram of Proposition 2.5 by a diagram chase.  $\square$

Although Corollary 2.6 gives a method to calculate  $\varphi(k)$ , in general it is not easy to have a formula for the dimension of  $Q_k$ . However we will see that it is possible to obtain such a formula in the monomial case.

### 3. THE MONOMIAL CASE

A rational monomial curve is such that the morphism  $f : \mathbb{P}^1 \rightarrow \mathbb{P}^s$  is given by homogeneous monomials of degree  $d$  in two variables. According to the above notations this is equivalent to say that  $T$  is generated by degree  $d$  monomials in the two variables  $x$  and  $y$  (see also Example 3 of [A-R2]). Let us fix, once for all, an ordered base of these monomials as follows:  $x^d, x^{d-1}y, x^{d-2}y^2, \dots, y^d$ ; in this way we establish a one to one correspondence in such a way that every monomial  $x^{d-i}y^i$  corresponds to the integer  $i$  for any  $i = 0, 1, \dots, d$ . We can always assume that  $T$  is generated by a set of these monomials and we can indicate these generators of  $T$  as the disjoint union of  $r$  subsets of  $[0, d]$ , each of them given by consecutive, increasing integers:  $T = \langle \alpha_1, \dots, \beta_1 \rangle \cup \langle \alpha_2, \dots, \beta_2 \rangle \cup \dots \cup \langle \alpha_r, \dots, \beta_r \rangle$  with  $\sum_{i=1}^r (\beta_i - \alpha_i + 1) = e + 1$ . Of course we can consider an analogous decomposition for  $\partial T$ , in this case we have subsets of  $[0, d-1]$ .

**Lemma 3.1.** *For any monomial  $x^p y^q$  ( $p \geq 1, q \geq 1$ ) we have:*

$$\begin{aligned}
& \partial \langle x^p y^q \rangle = \langle x^{p-1} y^q, x^p y^{q-1} \rangle \text{ and } \partial^{-1} \langle x^p y^q \rangle = 0; \\
& \partial^{-\nu} (\partial^\nu \langle x^p y^q \rangle) = \langle x^p y^q \rangle \text{ for any integer } \nu \geq 1; \\
& \dim(\partial^\nu \langle x^p y^q \rangle) = \nu + 1 \text{ if } \min\{p, q\} \geq \nu \geq 0 \text{ and } \dim(\partial^\nu \langle x^p y^q \rangle) = 0 \text{ if } \nu < 0.
\end{aligned}$$

*Proof.* They easily follow from the definitions of  $\partial$  and  $\partial^{-1}$ .  $\square$

Notation: in the sequel we will use this very useful notation: for any  $z \in \mathbb{Z}$  we will write  $\llbracket z \rrbracket$  for  $\max\{0, z\}$ . We have the following:

**Proposition 3.2.** *In the previous notation, let  $T$  be generated by  $e + 1$  distinct monomials among  $\{x^d, x^{d-1}y, x^{d-2}y^2, \dots, y^d\}$  with the decomposition  $T = \langle \alpha_1, \dots, \beta_1 \rangle \cup \langle \alpha_2, \dots, \beta_2 \rangle \cup \dots \cup \langle \alpha_r, \dots, \beta_r \rangle$ . Then:*

- i)  $T$  does not contain  $x^d, x^{d-1}y, xy^{d-1}, y^d$ , i.e.  $\alpha_1 \geq 2$  and  $\beta_r \leq d - 2$ ;*
- ii) if there exists some index  $i$  such that  $\alpha_{i+1} - \beta_i > 2$ , then we can split  $T = T' \oplus T''$ , in such a way that  $\partial^2 T = \partial^2 T' \oplus \partial^2 T''$ , by putting  $T' := \langle \alpha_1, \dots, \beta_1 \rangle \cup \langle \alpha_2, \dots, \beta_2 \rangle \cup \dots \cup \langle \alpha_i, \dots, \beta_i \rangle$  and  $T'' := \langle \alpha_{i+1}, \dots, \beta_{i+1} \rangle \cup \langle \alpha_{i+2}, \dots, \beta_{i+2} \rangle \cup \dots \cup \langle \alpha_r, \dots, \beta_r \rangle$ ;*
- iii) if no index as in ii) does exist, then  $\partial T = \langle \alpha_1 - 1, \dots, \beta_r \rangle$ ,  $\dim(\partial T) = \beta_r - \alpha_1 + 2$  and there exists a suitable monomial  $h$  such that  $\deg(h) = d + \beta_r - \alpha_1$  and  $\partial^{\beta_r - \alpha_1 + 1}(h) = \partial T$ ;*
- iv)  $T = \partial^{\beta_1 - \alpha_1}(m_1) \oplus \partial^{\beta_2 - \alpha_2}(m_2) \oplus \dots \oplus \partial^{\beta_r - \alpha_r}(m_r)$  for suitable monomials  $m_1, \dots, m_r$  such that  $\deg(m_i) = d + \beta_i - \alpha_i$ ,  $i = 1, \dots, r$ , and  $\dim(\partial^{-k}T) = \sum_{i=1}^r [\beta_i - \alpha_i - k + 1]$ .*

*Proof.* *i)* As  $\mathbb{P}(T) \cap \Gamma_d = \emptyset$ ,  $T$  cannot contain  $x^d$  and  $y^d$ : recall that  $\Gamma_d$  can be considered as the set of pure tensors of degree  $d$  in  $\mathbb{P}(S^d U)$ . Moreover the point  $(0 : 1 : 0 : \dots : 0)$  of  $\mathbb{P}(S^d U)$ , corresponding to  $x^{d-1}y$ , belongs to the tangent line to  $\Gamma_d$  at  $(1 : 0 : 0 : \dots : 0)$ , hence if  $T$  contains  $x^{d-1}y$  then  $\mathbb{P}(T)$  would intersect a tangent line to  $\Gamma_d$  and  $C$  would have a cusp. As we are assuming that  $C$  has at most ordinary singularities, this is not possible. The same argument runs also for  $xy^{d-1}$ .

*ii)* In our notation,  $\partial(\langle \alpha, \dots, \beta \rangle) = \langle \alpha - 1, \dots, \beta \rangle$  hence

$$\partial^2 T' = \langle \alpha_1 - 2, \dots, \beta_1 \rangle \cup \langle \alpha_2 - 2, \dots, \beta_2 \rangle \cup \dots \cup \langle \alpha_i - 2, \dots, \beta_i \rangle \text{ and}$$

$$\partial^2 T'' = \langle \alpha_{i+1} - 2, \dots, \beta_{i+1} \rangle \cup \langle \alpha_{i+2} - 2, \dots, \beta_{i+2} \rangle \cup \dots \cup \langle \alpha_r - 2, \dots, \beta_r \rangle,$$

as  $\alpha_{i+1} - 2 > \beta_i$  we have that  $\partial^2 T = \partial^2 T' \oplus \partial^2 T''$ .

*iii)* In this case, for any  $i = 1, \dots, r - 1$ , we have  $\alpha_{i+1} - \beta_i \leq 2$ ; as  $\alpha_{i+1} - \beta_i > 1$  by definition, we get  $\alpha_{i+1} - \beta_i = 2$ , i.e.  $\alpha_{i+1} - 2 = \beta_i$  for any  $i = 1, \dots, r - 1$ . hence  $\partial T = \langle \alpha_1 - 1, \dots, \beta_1 \rangle \cup \langle \alpha_2 - 1, \dots, \beta_2 \rangle \cup \dots \cup \langle \alpha_r - 1, \dots, \beta_r \rangle = \langle \alpha_1 - 1, \dots, \beta_r \rangle$  and  $\dim(\partial T) = \beta_r - \alpha_1 + 2$  as  $\partial T$  is generated by  $\beta_r - \alpha_1 + 2$  independent monomials:  $x^{d-\alpha_1}y^{\alpha_1-1}, \dots, x^{d-1-\beta_r}y^{\beta_r}$ . Moreover if we consider  $h := x^{d-\alpha_1}y^{\beta_r}$  we have that  $\partial^{\beta_r - \alpha_1 + 1}(h) = \partial T$ .

*iv)* The decomposition for  $T$  is a reformulation of our assumption; the degree of monomials  $m_i$  follows by arguing as in *iii)* for the degree of  $h$ ; the dimension of  $\partial^{-k}T = \partial^{\beta_1 - \alpha_1 - k}(m_1) \oplus \partial^{\beta_2 - \alpha_2 - k}(m_2) \oplus \dots \oplus \partial^{\beta_r - \alpha_r - k}(m_r)$  is the sum of the dimensions of  $\partial^{\beta_i - \alpha_i - k}(m_i)$  for  $i = 1, \dots, r$ , hence the last formula follows from Lemma 3.1.  $\square$

**Remark 3.1.** *From Proposition 3.2 it follows that to calculate  $\varphi(k)$  for monomial curves it suffices to have a formula for any vector space  $T$  such that condition *iii)* of such Proposition holds.*

By Remark 3.1, to solve our problem it suffices to consider vector spaces  $T$  satisfying condition *iii)* of Proposition 3.2. Therefore, from now on we will assume that  $T = \langle \alpha_1, \dots, \beta_1 \rangle \cup \langle \alpha_2, \dots, \beta_2 \rangle \cup \dots \cup \langle \alpha_r, \dots, \beta_r \rangle$ , with  $\alpha_1 \geq 2$ ,  $\beta_r \leq d - 2$ ,  $\alpha_i = \beta_{i-1} + 2$  for any  $i = 2, \dots, r$  and  $\partial T = \langle \alpha_1 - 1, \dots, \beta_r \rangle$ .

Note that, in this case, it is more useful to define  $T$  as follows:

**Definition 3.1.** *Let us choose an integer  $\lambda \geq 2$ ,  $\lambda$  monomials of degree  $d - 1$ , which are consecutive with respect to the powers of  $y$ , and a partition  $\gamma_1, \gamma_2, \dots, \gamma_r$  of*

the integer  $\lambda$  such that  $r \geq 2$ ,  $\gamma_i \geq 2$ ,  $\gamma_1 + \gamma_2 + \dots + \gamma_r = \lambda$ . Let us consider a vector space  $T \subset S^d U$  such that  $T = \partial^{\gamma_1-2}(m_1) \oplus \dots \oplus \partial^{\gamma_r-2}(m_r)$  and  $\partial T = \partial^{\lambda-1}(h)$ , where  $m_i$  are suitable monomials of degree  $d+2-\gamma_i$ , and  $h$  is a suitable monomial of degree  $d+\lambda-2$ , determined as in iii) and iv) of Proposition 3.2 (in particular  $\partial T$  is generated by the above  $\lambda$  consecutive monomials). Then we say that  $T$  is special,  $\lambda$  will be its height,  $h$  its apex (of degree  $d+\lambda-2$ ) and  $(\gamma_1, \gamma_2, \dots, \gamma_r)$  the associated partition of  $\lambda$ .

**Proposition 3.3.** *Let  $T \subset S^d U$  be a special vector space of height  $\lambda$ , apex  $h$  and partition  $(\gamma_1, \gamma_2, \dots, \gamma_r)$ . Then:*

- i)  $T = \langle \alpha_1, \dots, \beta_1 \rangle \cup \langle \alpha_2, \dots, \beta_2 \rangle \cup \dots \cup \langle \alpha_r, \dots, \beta_r \rangle$ , with  $\alpha_1 \geq 2$ ,  $\beta_r \leq d-2$ ,  $\alpha_i = \beta_{i-1} + 2$  for any  $i = 2, \dots, r$ ;
- ii)  $\partial T = \langle \alpha_1 - 1, \dots, \beta_r \rangle$  and  $\lambda = \beta_r - \alpha_1 + 2 = \dim(\partial T)$ ;
- iii) for any  $k \geq 2$ ,  $\varphi(k) = \dim \partial^{-k} T + \dim \partial^{-k+1} \partial T - \dim(Q_k) = \dim \partial^{-k} T + \dim \partial^{\lambda-k}(h) - q(k)$ , where:

$$\dim \partial^{-k} T = \sum_{i=1}^r [\gamma_i - 1 - k];$$

$$\dim \partial^{\lambda-k}(h) = [\lambda - k + 1];$$

$$q(k) = \dim \left[ \frac{p_{k-1}(\partial^{\lambda-k}(h))}{D_k(S^k U \otimes T) \cap p_{k-1}(\partial^{\lambda-k}(h))} \right] \text{ for } k \in [2, \lambda] \text{ and } q(k) = 0 \text{ for } k \geq \lambda + 1;$$

- iv) the number of integers  $c_i$  (see § 2) such that  $c_i = 0$  is  $d - 2 - \lambda$ .

*Proof.* i) and ii) are merely reformulations of the assumptions on  $T$ . iii) follows from Corollary 2.6 and from Lemma 3.1.

To prove iv) we use Theorem 2.3. In our case  $\partial^2 T = \langle \alpha_1 - 2, \dots, \beta_r \rangle$  so that  $\dim(\partial^2 T) = \beta_r - (\alpha_1 - 2) + 1 = \lambda + 1$ . Note that  $d - 2 - \lambda \geq 0$  because  $\alpha_1 \geq 2$ ,  $\beta_r \leq d - 2$ , hence  $\beta_r - \alpha_1 \leq d - 4$  and  $\lambda \leq d - 2$ .  $\square$

From Proposition 3.3 it follows that, to solve our problem, it suffices to calculate  $q(k)$  for  $k \in [2, \lambda]$  for any special  $T$ .

#### 4. REDUCTION TO A COMBINATORIAL PROBLEM

The aim of this section is to determine a formula for calculating  $q(k)$  for  $k \in [2, \lambda]$  and for any special  $T \subset S^d U$ , of height  $\lambda$ , apex  $h$  of degree  $d + \lambda - 2$  and partition  $(\gamma_1, \gamma_2, \dots, \gamma_r)$ .

As  $q(k) = \dim \left[ \frac{p_{k-1}(\partial^{\lambda-k}(h))}{D_k(S^k U \otimes T) \cap p_{k-1}(\partial^{\lambda-k}(h))} \right]$ , we need a description of  $p_{k-1}(\partial^{\lambda-k}(h))$ . For any  $\partial^{\lambda-k}(h)$  we want to fix, once for all, a set of independent generators. Note that  $\partial T = \partial^{\lambda-1}(h) = \langle \partial_y^{\lambda-1}(h), \partial_x \partial_y^{\lambda-2}(h), \dots, \partial_x^{\lambda-1}(h) \rangle$  where the generators are monomials of degree  $d-1$ , then  $\partial^{\lambda-k}(h) = \langle \partial_y^{\lambda-k}(h), \partial_x \partial_y^{\lambda-k-1}(h), \dots, \partial_x^{\lambda-k}(h) \rangle$  and we can choose this set of  $\lambda - k + 1$  independent ordered generators (all of them being monomials of degree  $d + k - 2$ ) and, in the sequel, a generator for  $\partial^{\lambda-k}(h)$  will be always an element of this set. Recall that  $p_{k-1}$  is injective, so that we get a fixed set of independent generators also for  $p_{k-1}(\partial^{\lambda-k}(h))$ .

For any  $j = 0, \dots, \lambda - k$

$$p_{k-1}(\partial_x^j \partial_y^{\lambda-k-j}(h)) = \frac{(d+1)!}{(d+k-2)!} \sum_{\mu=0}^{k-1} \binom{k-1}{\mu} x^{k-1-\mu} y^\mu \otimes \partial_x^{k-1-\mu+j} \partial_y^{\mu+\lambda-k-j}(h)$$

so that, all  $\lambda - k + 1$  independent generators of  $p_{k-1}(\partial_x^j \partial_y^{\lambda-k-j}(h))$  are the sum of the tensor product of the monomials of the fixed polynomial  $\frac{(d+1)!}{(d+k-2)!} \sum_{\mu=0}^{k-1} \binom{k-1}{\mu} x^{k-1-\mu} y^\mu$

(i.e. independent from  $j$ ) with a set of  $k$  consecutive independent generators of  $\partial T$ , consecutive with respect our fixed order. More precisely, for  $j = 0$  we have the first set of  $k$  generators of  $\partial T$   $\{\partial_y^{\lambda-1}(h), \partial_x \partial_y^{\lambda-2}(h), \dots, \partial_x^{k-1} \partial_y^{\lambda-k}(h)\}$ , although in the reverse order with respect to the order we have chosen.

For  $j = 1$  we have the second set:  $\{\partial_x \partial_y^{\lambda-2}(h), \partial_x^2 \partial_y^{\lambda-3}(h), \dots, \partial_x^{k-2} \partial_y^{\lambda-k+1}(h)\}$  although in the reverse order with respect to the order we have chosen, ... , for  $j = \lambda - k$  we have the last set:  $\{\partial_x^{\lambda-k} \partial_y^{k-1}(h), \partial_x^{\lambda-k+1} \partial_y^{k-2}(h), \dots, \partial_x^{\lambda-1}(h)\}$  although in the reverse order with respect to the order we have chosen.

Now we want to decide wether a generator of  $p_{k-1}(\partial^{\lambda-k}(h))$  belongs to  $D_k(S^k U \otimes T)$  or not. As  $D_k(S^k U \otimes T) \subseteq S^{k-1} U \otimes \partial T$ , the generic element of  $D_k(S^k U \otimes T)$  has the form

$$g_1 \otimes \partial_y^{\lambda-1}(h) + g_2 \otimes \partial_x \partial_y^{\lambda-2}(h) + \dots + g_\lambda \otimes \partial_x^{\lambda-1}(h)$$

for suitable polynomials  $g_i \in S^{k-1} U$ . As each of the  $\lambda - k + 1$  generators  $p_{k-1}(\partial_x^j \partial_y^{\lambda-k-j}(h))$  involves only  $k$  generators of  $\partial T$ , to establish whether a particular generator of  $p_{k-1}(\partial^{\lambda-k}(h))$  belongs to  $D_k(S^k U \otimes T)$  or not, we have to consider only  $k$  generators of  $\partial T$ . For instance, when  $j = 0$ ,  $p_{k-1}(\partial_y^{\lambda-k}(h)) \in D_k(S^k U \otimes T)$  if and only if there exist polynomials  $g_1, \dots, g_k$  such that  $g_{1+k-1-\mu} = \frac{(d+1)!}{(d+k-2)!} \binom{k-1}{\mu} x^{k-1-\mu} y^\mu$  for  $\mu = 0, \dots, k-1$  up to a non zero common constant; when  $j = 1$ ,  $p_{k-1}(\partial_x \partial_y^{\lambda-k-1}(h)) \in D_k(S^k U \otimes T)$  if and only if there exist polynomials  $g_2, \dots, g_{k+1}$  such that  $g_{2+k-1-\mu} = \frac{(d+1)!}{(d+k-2)!} \binom{k-1}{\mu} x^{k-1-\mu} y^\mu$  for  $\mu = 0, \dots, k-1$  up to a non zero common constant and so on.

Of course, polynomials  $g_1, \dots, g_\lambda$  are not generic; they can be divided into  $r$  sets according to the partition  $\gamma_1, \gamma_2, \dots, \gamma_r$ ; more precisely:

$$\begin{aligned} g_\lambda &= -\partial_y f_{e+1} \\ g_{\lambda-1} &= \partial_x f_{e+1} - \partial_y f_e \\ g_{\lambda-2} &= \partial_x f_e - \partial_y f_{e-1} \\ &\dots \\ g_{\lambda-(\gamma_r-1)} &= \partial_x f_{e+1-(\gamma_r-2)} \\ &\dots \\ g_{\lambda-(\gamma_r-1)-1} &= -\partial_y f_{e+1-(\gamma_r-2)-1} \\ g_{\lambda-(\gamma_r-1)-2} &= \partial_x f_{e+1-(\gamma_r-2)-1} - \partial_y f_{e+1-(\gamma_r-2)-2} \\ &\dots \\ g_{\lambda-(\gamma_r-1)-1-(\gamma_{r-1}-1)} &= \partial_x f_{e+1-(\gamma_r-2)-1-(\gamma_{r-1}-2)} \\ &\dots \\ &\dots \\ g_{\gamma_1} &= -\partial_y f_{\gamma_1-1} \\ g_{\gamma_1-1} &= \partial_x f_{\gamma_1-1} - \partial_y f_{\gamma_1-2} \\ &\dots \\ g_1 &= \partial_x f_1 \end{aligned} \tag{*}$$

where now  $\{f_\zeta | \zeta = 1, \dots, e+1\}$  are generic polynomials of degree  $k$ . Recall that  $\dim(T) = e+1 = \lambda - r$  and that we have reversed our ordering for the  $\lambda$  generators of  $\partial T$ . Let us call  $(*)$  the above list of  $\lambda$  polynomials.

For any generic polynomial  $f \in S^k U$ , we can write:  $f = \sum_{i=0}^k \binom{k}{i} a_i x^{k-i} y^i$  with  $a_i \in \mathbb{C}$ , hence:



$$\begin{aligned}\partial_x f &= k \sum_{i=0}^k \binom{k}{i} a_i (k-i) x^{k-i-1} y^i = k \sum_{i=0}^{k-1} \binom{k-1}{i} a_i x^{k-1-i} y^i \\ \partial_y f &= k \sum_{i=0}^k \binom{k}{i} a_i i x^{k-i} y^{i-1} = k \sum_{i=0}^{k-1} \binom{k-1}{i} a_{i+1} x^{k-1-i} y^i.\end{aligned}$$

It follows that the generic polynomial of  $(*)$  is of the following type:

$$\begin{aligned}\partial_x f_{\nu+1} - \partial_y f_{\nu} &= k \sum_{i=0}^{k-1} \binom{k-1}{i} a_{\nu+1,i} x^{k-1-i} y^i - k \sum_{i=0}^{k-1} \binom{k-1}{i} a_{\nu,i+1} x^{k-1-i} y^i = \\ &= k \sum_{i=0}^{k-1} \binom{k-1}{i} (a_{\nu+1,i} - a_{\nu,i+1}) x^{k-1-i} y^i.\end{aligned}$$

**Remark 4.1.** The above expression is true for all polynomial of  $(*)$  if we remind that for some values of  $\nu = 1, \dots, \lambda$  we have  $\partial_y f_{\nu} = 0$  or  $\partial_x f_{\nu+1} = 0$  (more precisely  $\partial_y f_{\nu} = 0$  for  $\nu = 1, \gamma_1 + 1, \gamma_1 + \gamma_2 + 1, \dots, \gamma_1 + \gamma_2 + \dots + \gamma_{r-1} + 1$ ;  $\partial_x f_{\nu+1} = 0$  for  $\nu + 1 = \gamma_1, \gamma_1 + \gamma_2, \dots, \lambda$ , but this not relevant in our context).

**Definition 4.1.** Let us call  $\mathcal{P}$  any element of the partition, i.e.  $\mathcal{P}$  is a subset of  $\{1, 2, \dots, \lambda\}$  given by consecutive integers according to the partition  $\gamma_1 + \gamma_2 + \dots + \gamma_r = \lambda$ . As we have seen above, any generator  $G$  of  $p_{k-1}(\partial^{\lambda-k}(h))$  involves only a precise set of  $k$  generators of  $\partial T$ . Such set is in a 1 : 1 correspondence with a set  $G_s$  of  $k$  consecutive integers belonging to  $[1, \lambda]$ . We will say that a generator  $G$  of  $p_{k-1}(\partial^{\lambda-k}(h))$  covers an element  $\mathcal{P}$  of the partition if  $G_s$  contains  $\mathcal{P}$ . Such generators will be called covering generators.

The aim of this Section is to prove the following:

**Theorem 4.1.** Let  $T \subset S^d U$  be a special vector space of heighth  $\lambda$ , apex  $h$  and partition  $(\gamma_1, \gamma_2, \dots, \gamma_r)$ , then, for any  $k \in [2, \lambda]$ ,  $q(k)$  is the number of covering generators of  $p_{k-1}(\partial^{\lambda-k}(h))$ .

Before proving Theorem 4.1 we need the following:

**Lemma 4.2.** Let  $z_1, z_2, \dots, z_q$  complex variables ( $q \geq 2$ ) and let us consider a linear system of  $q-1$  equations as follows:

$$\varepsilon z_1 - z_2 = w_1$$

$$z_2 - z_3 = w_2$$

$$z_3 - z_4 = w_3$$

.....

$$z_{q-1} - \eta z_q = w_{q-1}$$

where  $\varepsilon, \eta \in \mathbb{C}$  and  $\mathbf{w} := (w_1, w_2, \dots, w_{q-1}) \in \mathbb{C}^{q-1}$ . Then:

i) if  $\varepsilon \neq 0$  and  $\eta \neq 0$ , the linear system has always solutions, more precisely  $\infty^1$  solutions;

ii) if  $\varepsilon = 0$  and  $\eta \neq 0$ , or  $\varepsilon \neq 0$  and  $\eta = 0$ , the linear system has always a unique solution, hence it has a non zero solution if and only if  $\mathbf{w} \neq \mathbf{0}$ ;

iii) if  $\varepsilon = \eta = 0$ , the linear system has a solution if and only if  $\sum_{i=1}^{q-1} w_i = 0$ , in this case it has a unique solution; hence it has a non zero solution if and only if  $\mathbf{w} \neq \mathbf{0}$  but  $\sum_{i=1}^{q-1} w_i = 0$ .

*Proof.* Let us consider the associated  $(q-1, q)$  matrix:

$$\begin{bmatrix} \varepsilon & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & -1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & -\eta \end{bmatrix}.$$

In case *i*) we have a linear map  $\mathbb{C}^q \rightarrow \mathbb{C}^{q-1}$  always of maximal rank. In case *ii*) we have a linear map  $\mathbb{C}^{q-1} \rightarrow \mathbb{C}^{q-1}$  of maximal rank. In case *iii*) we have a linear map  $\mathbb{C}^{q-2} \rightarrow \mathbb{C}^{q-1}$  whose associated matrix has rank  $q-2$  and it is easy to see that the linear system has a (unique) solution if and only if  $\sum_{i=1}^{q-1} w_i = 0$ .  $\square$

*Proof.* (of Theorem 4.1) Step 1.

Let  $\Delta$  be any generator of  $p_{k-1}(\partial^{\lambda-k}(h))$ . We have that  $\Delta \in D_k(S^k U \otimes T)$  if and only if there is a non zero solution of the corresponding subset of  $k$  equations (in the unknowns  $a_{\nu,i}$ ) extracted from the big set (\*). More precisely: the first generator involves equations  $1, \dots, k$  (counting from below), the second generators involves equations  $2, \dots, k+1, \dots$ , the last generator involves equations  $\lambda-k+1, \dots, \lambda$ . The  $k$  equations for  $\Delta$  can be indexed by  $\mu = 0, \dots, k-1$  as follows:

$$\begin{aligned} E(\Delta, 0); & k \sum_{i=0}^{k-1} \binom{k-1}{i} (a_{\nu+1,i} - a_{\nu,i+1}) x^{k-1-i} y^i = \rho'_\Delta \frac{(d+1)!}{(d+k-2)!} \binom{k-1}{0} x^{k-1} \\ E(\Delta, 1); & k \sum_{i=0}^{k-1} \binom{k-1}{i} (a_{\nu,i} - a_{\nu-1,i+1}) x^{k-1-i} y^i = \rho'_\Delta \frac{(d+1)!}{(d+k-2)!} \binom{k-1}{1} x^{k-2} y \\ \dots\dots\dots & \\ E(\Delta, \mu); & k \sum_{i=0}^{k-1} \binom{k-1}{i} (a_{\nu+1-\mu,i} - a_{\nu-\mu,i+1}) x^{k-1-i} y^i = \rho'_\Delta \frac{(d+1)!}{(d+k-2)!} \binom{k-1}{\mu-1} x^{k-\mu} y^{\mu-1} \\ \dots\dots\dots & \\ E(\Delta, k-1); & k \sum_{i=0}^{k-1} \binom{k-1}{i} (a_{\nu+2-k,i} - a_{\nu+1-k,i+1}) x^{k-1-i} y^i = \rho'_\Delta \frac{(d+1)!}{(d+k-2)!} \binom{k-1}{k-1} y^{k-1}. \end{aligned}$$

Of course, for some  $\nu$ , the equations  $E(\Delta, \mu)$  can be different by Remark 4.1. As  $k$  and  $d$  are fixed we can simplify a little bit any equation by putting  $\rho_\Delta = (\rho'_\Delta \frac{(d+1)!}{(d+k-2)!}) (\frac{1}{k})$ . Note that  $\rho_\Delta \neq 0$  and it depends only on  $\Delta$ .

The set of  $k$  equations corresponding to  $\Delta$  can be subdivided into  $t_\Delta$  linear systems  $\mathcal{L}_1, \dots, \mathcal{L}_{t_\Delta}$ , where  $t_\Delta \geq 1$  is the number of the elements  $\mathcal{P}$  of the partition such that the intersection of  $\mathcal{P}$  with the  $k$  consecutive integers corresponding to  $\Delta$  is not empty. Moreover anyone of these linear systems can be further subdivided into smaller linear subsystems by considering the variables  $a_{\nu,i}$  for which the difference  $|\nu - i|$  is constant. Note the following facts:

- anyone of these linear subsystems belongs to one of the types *i*), *ii*) or *iii*) considered by Lemma 4.2;
- anyone of these linear subsystems involves different variables;
- in any  $\mathcal{L}_j$  ( $j = 1, \dots, t_\Delta$ ) there is one and only one subsystem  $\overline{\mathcal{L}}_j$  which is not homogeneous;
- to solve the linear system  $\overline{\mathcal{L}}_j$  means to find the fibre of a suitable linear map over a vector of type  $(\rho_\Delta, \rho_\Delta, \dots, \rho_\Delta)$ .

As the common factor  $\rho_\Delta \neq 0$  depends only on  $\Delta$ , we have that  $\Delta \in D_k(S^k U \otimes T)$  if and only if all non homogeneous linear systems  $\overline{\mathcal{L}}_1, \dots, \overline{\mathcal{L}}_{t_\Delta}$  have a non zero solution (if a system  $\overline{\mathcal{L}}_j$  has only the zero solution then all of them have only the zero solution). Viceversa  $\Delta \notin D_k(S^k U \otimes T)$  if and only if, among  $\overline{\mathcal{L}}_1, \dots, \overline{\mathcal{L}}_{t_\Delta}$ , there exists at least one linear system having only the zero solution. By Lemma 4.2 this

is possible if and only if there exists at least a linear system  $\overline{\mathcal{L}}_j$  of type *iii*) and this is true if and only if  $\Delta$  covers at least an element  $\mathcal{P}$  of the partition.

Example. To clarify the above argument, let us describe an example in which  $k = 4$ ,  $\lambda = \gamma_1 + \gamma_2 + \gamma_3 = 3 + 3 + 4 = 10$  and  $\Delta$  covers the element  $\mathcal{P} \longleftrightarrow \gamma_2$  of the partition, with  $t_\Delta = 2$ . To simplify notations, let us call the variables of (\*) with different letters. We have the following diagram, where  $g_1, \dots, g_\lambda$  are obtained multiplying the elements of the first four columns with the monomials at the top and the rows containing  $\rho_\Delta$  correspond to the equations  $E(\Delta, 0), \dots, E(\Delta, 3)$  in this case:

$$\begin{array}{cccccc}
 x^3 & 3x^2y & 3xy^2 & y^3 & & \\
 -a_1 & -a_2 & -a_3 & -a_4 & & \\
 a_0 - b_1 & a_1 - b_2 & a_2 - b_3 & a_3 - b_4 & & \\
 b_0 - c_1 & b_1 - c_2 & b_2 - c_3 & b_3 - c_4 & & \\
 c_0 & c_1 & c_2 & c_3 & \rho_\Delta x^3 & \\
 -d_1 & -d_2 & -d_3 & -d_4 & \rho_\Delta 3x^2y & \\
 d_0 - e_1 & d_1 - e_2 & d_2 - e_3 & d_3 - e_4 & \rho_\Delta 3xy^2 & \\
 e_0 & e_1 & e_2 & e_3 & \rho_\Delta y^3 & \\
 -p_1 & -p_2 & -p_3 & -p_4 & & \\
 p_0 - q_1 & p_1 - q_2 & p_2 - q_3 & p_3 - q_4 & & \\
 q_0 & q_1 & q_2 & q_3 & & 
 \end{array}$$

and  $\Delta \in D_3(S^3U \otimes T)$  if and only if there exist solutions, with  $\rho_\Delta \neq 0$ , of the linear systems  $\mathcal{L}_1$  and  $\mathcal{L}_2$  implied by the following equations:

$$\begin{aligned}
 \mathcal{L}_1) \quad & c_0x^3 + c_13x^2y + c_23xy^2 + c_3y^3 = \rho_\Delta x^3 \\
 \mathcal{L}_2) \quad & -d_1x^3 - d_23x^2y - d_33xy^2 - d_4y^3 = \rho_\Delta 3x^2y \\
 & (d_0 - e_1)x^3 + (d_1 - e_2)3x^2y + (d_2 - e_3)3xy^2 + (d_3 - e_4)y^3 = \rho_\Delta 3xy^2 \\
 & e_0x^3 + e_13x^2y + e_23xy^2 + e_3y^3 = \rho_\Delta y^3.
 \end{aligned}$$

By considering the linear subsystems for which  $|\nu - i|$  is constant, (i.e. those involving variables on the descending diagonals  $\searrow$  of  $\mathcal{L}_1 \cup \mathcal{L}_2$ , note that this is a partition of the variables) we have that there exists only one non homogeneous subsystem  $\overline{\mathcal{L}}_2 \subset \mathcal{L}_2$  :

$$\begin{aligned}
 -d_2 &= \rho_\Delta \\
 d_2 - e_3 &= \rho_\Delta \\
 e_3 &= \rho_\Delta
 \end{aligned}$$

implying  $\rho_\Delta = 0$ . Hence  $\Delta \notin D_3(S^3U \otimes T)$ . Note that, if we change generator, for instance by shifting the left column of the diagram to the top, this generator is not a covering one and the corresponding  $\mathcal{L}_1$  and  $\mathcal{L}_2$  become:

$$\begin{aligned}
 \mathcal{L}_1) \quad & (b_0 - c_1)x^3 + (b_1 - c_2)3x^2y + (b_2 - c_3)3xy^2 + (b_3 - c_4)y^3 = \rho_\Delta x^3. \\
 & c_0x^3 + c_13x^2y + c_23xy^2 + c_3y^3 = \rho_\Delta 3x^2y \\
 \mathcal{L}_2) \quad & -d_1x^3 - d_23x^2y - d_33xy^2 - d_4y^3 = \rho_\Delta 3xy^2 \\
 & (d_0 - e_1)x^3 + (d_1 - e_2)3x^2y + (d_2 - e_3)3xy^2 + (d_3 - e_4)y^3 = \rho_\Delta y^3.
 \end{aligned}$$

There are only the following non homogeneous subsystems  $\overline{\mathcal{L}}_i \subset \mathcal{L}_i$  :

$$\begin{aligned}
 \overline{\mathcal{L}}_1) \quad & b_0 - c_1 = \rho_\Delta \\
 & c_1 = \rho_\Delta \\
 \overline{\mathcal{L}}_2) \quad & -d_3 = \rho_\Delta \\
 & d_3 - e_4 = \rho_\Delta
 \end{aligned}$$

having solutions with  $\rho_\Delta \neq 0$ , and all other subsystems have solutions, hence the generator belongs to  $D_3(S^3U \otimes T)$ .

Step 2. Let  $\Delta_1, \dots, \Delta_q$  be the set of covering generators of  $p_{k-1}(\partial^{\lambda-k}(h))$ . To get the proof of Theorem 4.1 we have to show that  $\{\Delta_1, \dots, \Delta_q\}$  is a base for  $Q_k$ , i.e. that no non zero element  $\chi_1\Delta_1 + \dots + \chi_q\Delta_q$  ( $\chi_i \in \mathbb{C}$ ) belongs to  $D_k(S^kU \otimes T)$ . The argument of Step 1 implies that this is obviously true if the generators cover different  $\mathcal{P}$  : assuming that  $\chi_1\Delta_1 + \dots + \chi_q\Delta_q \in D_k(S^kU \otimes T)$ , for any  $i = 1, \dots, q$  we would get a non homogeneous linear system implying  $\chi_i = 0$  and these linear systems would involve different variables. But this is true also when two (or more) generators cover the same  $\mathcal{P}$  because the sets of consecutive covered integers cannot coincide for distinct generators, hence the variables involved by the non homogeneous subsystems are always different. We think that an example is sufficient to explain why.

Let us consider the previous example and let us assume that there exists another generator  $\Delta'$  covering the same  $\mathcal{P} \longleftrightarrow \gamma_2$ . There is only one possibility, described by the following diagram, using the same notations as in the previous one:

$$\begin{array}{ccccccc}
 x^3 & 3x^2y & 3xy^2 & y^3 & & & \\
 -a_1 & -a_2 & -a_3 & -a_4 & & & \\
 a_0 - b_1 & a_1 - b_2 & a_2 - b_3 & a_3 - b_4 & & & \\
 b_0 - c_1 & b_1 - c_2 & b_2 - c_3 & b_3 - c_4 & & & \\
 c_0 & c_1 & c_2 & c_3 & & & \\
 -d_1 & -d_2 & -d_3 & -d_4 & \rho_\Delta x^3 & & \\
 d_0 - e_1 & d_1 - e_2 & d_2 - e_3 & d_3 - e_4 & \rho_\Delta 3x^2y + \rho_{\Delta'} x^3 & & \\
 e_0 & e_1 & e_2 & e_3 & \rho_\Delta 3xy^2 + \rho_{\Delta'} 3x^2y & & \\
 -p_1 & -p_2 & -p_3 & -p_4 & \rho_\Delta y^3 + \rho_{\Delta'} 3xy^2 & & \\
 p_0 - q_1 & p_1 - q_2 & p_2 - q_3 & p_3 - q_4 & \rho_{\Delta'} y^3 & & \\
 q_0 & q_1 & q_2 & q_3 & & & 
 \end{array}$$

note that, as  $k$  and  $d$  are fixed, we can use coefficients  $\rho_\Delta$  and  $\rho_{\Delta'}$  as in Step 1 and  $\chi_\Delta = 0$  if and only if  $\rho_\Delta = 0$ ,  $\chi_{\Delta'} = 0$  if and only if  $\rho_{\Delta'} = 0$ .

We get a set of five equations, three of them for  $\mathcal{P} \longleftrightarrow \gamma_2$  :

$$\begin{aligned}
 -d_1x^3 - d_23x^2y - d_33xy^2 - d_4y^3 &= \rho_\Delta 3x^2y + \rho_{\Delta'} x^3 \\
 (d_0 - e_1)x^3 + (d_1 - e_2)3x^2y + (d_2 - e_3)3xy^2 + (d_3 - e_4)y^3 &= \rho_\Delta 3xy^2 + \rho_{\Delta'} 3x^2y \\
 e_0x^3 + e_13x^2y + e_23xy^2 + e_3y^3 &= \rho_\Delta y^3 + \rho_{\Delta'} 3xy^2
 \end{aligned}$$

and we have:

$$\begin{aligned}
 -d_2 &= \rho_\Delta \\
 d_2 - e_3 &= \rho_\Delta \\
 e_3 &= \rho_\Delta \\
 \text{implying } \rho_\Delta &= 0, \text{ as in Step 1, and} \\
 -d_1 &= \rho_{\Delta'} \\
 d_1 - e_2 &= \rho_{\Delta'} \\
 e_2 &= \rho_{\Delta'} \\
 \text{implying } \rho_{\Delta'} &= 0.
 \end{aligned}$$

□

## 5. A COMBINATORIAL FORMULA

As we have seen in the previous Sections to compute  $\varphi(k)$  it suffices to have a formula for calculating  $q(k)$  for every special vector space  $T \subset S^dU$  and Theorem 4.1 says that we have to compute the number of covering generators among the generators of  $p_{k-1}(\partial^{\lambda-k}(h))$ . It is a combinatorial problem because:

-  $T$  is determined by the partition  $\gamma_1 + \gamma_2 + \dots + \gamma_r = \lambda$  ( $r \geq 2, \gamma_i \geq 2$ ),

- every generator of  $p_{k-1}(\partial^{\lambda-k}(h))$  ( $k \in [2, \lambda]$ ) involves a set of  $k$  consecutive integers  $[1, \dots, k]$ ,  $[2, \dots, k+1]$ ,  $\dots$ ,  $[\lambda-k+1, \lambda]$ , subsets of  $[1, \lambda]$ ,
- we have to compute the number of such subsets containing at least an element  $\mathcal{P}$  of the partition.

Although this is a very simple problem, we have found nothing about it in the literature.

Let us put the above combinatorial problem in the correct perspective. Let  $\lambda$  be a positive integer,  $\lambda \geq 2$ . Let  $\gamma_1 + \gamma_2 + \dots + \gamma_r = \lambda$  ( $r \geq 1, \gamma_i \geq 1$ ) be a partition of  $\lambda$  giving rise to (infinitely many) subsets of  $\mathbb{Z}$  in the following way:  $[a_1, b_1] \cup [a_2, b_2] \cup \dots \cup [a_r, b_r]$  with  $\gamma_i = b_i - a_i + 1$  for  $i = 1, \dots, r$  and  $a_i = b_{i-1} + 1$  for  $i = 2, \dots, r$ . We will indicate such a partition as  $(\gamma_1, \gamma_2, \dots, \gamma_r)$ . Let us fix a positive integer  $k \geq 1$  and for any  $j \in \mathbb{Z}$  let us consider the subset  $\Lambda_j(k) := [j+1, j+k] \subset \mathbb{Z}$ . We say that  $\Lambda_j(k)$  is a *covering set* for the partition  $(\gamma_1, \gamma_2, \dots, \gamma_r)$  if there exists at least an index  $i = 1, \dots, r$  such that  $\Lambda_j(k) \supseteq [a_i, b_i]$ . Let us define the following functions:

$$\begin{aligned} \tilde{q}(k, \gamma_1, \gamma_2, \dots, \gamma_r) &= \#\{\Lambda_j(k) \mid \Lambda_j(k) \text{ is a covering set for } (\gamma_1, \gamma_2, \dots, \gamma_r)\} \\ q(k, \gamma_1, \gamma_2, \dots, \gamma_r) &= \#\{\Lambda_j(k) \mid \Lambda_j(k) \text{ is a covering set for } (\gamma_1, \gamma_2, \dots, \gamma_r) \text{ and } \Lambda_j(k) \subseteq [a_1, b_r]\}. \end{aligned}$$

Note that the above functions depend only from  $k$  and the partition  $(\gamma_1, \gamma_2, \dots, \gamma_r)$ , but they do not depend from the values of  $a_i$  and  $b_i$ .

Obviously, for a special vector space  $T \subset S^k U$  having height  $\lambda$  and apex  $h$ , and for which the partition of  $\lambda$  is  $(\gamma_1, \gamma_2, \dots, \gamma_r)$ , we have that, for any  $k \in [2, \lambda]$ ,  $q(k) = q(k, \gamma_1, \gamma_2, \dots, \gamma_r)$ . Hence we solve our problem by giving a formula for calculating  $q(k, \gamma_1, \gamma_2, \dots, \gamma_r)$ .

**Lemma 5.1.** *For the functions  $\tilde{q}(k, \gamma_1, \gamma_2, \dots, \gamma_r)$  we have:*

- i)  $\tilde{q}(k, \gamma_1) = \llbracket k - \gamma_1 + 1 \rrbracket$ ;
  - ii) for any  $i = 1, \dots, r-1$
- $$\tilde{q}(k, \gamma_1, \dots, \gamma_i, \gamma_{i+1}, \dots, \gamma_r) = \tilde{q}(k, \gamma_1, \dots, \gamma_i) + \tilde{q}(k, \gamma_{i+1}, \dots, \gamma_r) - \tilde{q}(k, \gamma_i + \gamma_{i+1}).$$

*Proof.* i)  $\Lambda_j(k) \supseteq [a_1, b_1] = [a_1, a_1 + \gamma_1 - 1]$  if and only if  $k \geq \gamma_1$ ,  $j+1 \leq a_1$  and  $j+k \geq b_1$ , i.e.  $j \in [b_1 - k, a_1 - 1]$ , hence, if  $k \geq \gamma_1$ ,  $\tilde{q}(k, \gamma_1) = a_1 - 1 - (b_1 - k) + 1 = k - \gamma_1 + 1$ . Then  $\tilde{q}(k, \gamma_1) = \llbracket k - \gamma_1 + 1 \rrbracket$ .

ii)  $\Lambda_j(k)$  covers  $(\gamma_1, \gamma_2, \dots, \gamma_r)$  if and only if it covers  $(\gamma_1, \dots, \gamma_i)$  or it covers  $(\gamma_{i+1}, \dots, \gamma_r)$  or both. In this last case  $\Lambda_j(k)$  necessarily covers the set  $[a_i, b_i] \cup [a_{i+1}, b_{i+1}] = [a_i, b_{i+1}]$  with  $b_{i+1} - a_i + 1 = \gamma_i + \gamma_{i+1}$ , hence to compute correctly  $\tilde{q}(k, \gamma_1, \dots, \gamma_r)$  we have to subtract  $\tilde{q}(k, \gamma_i + \gamma_{i+1})$  from the sum  $\tilde{q}(k, \gamma_1, \dots, \gamma_i) + \tilde{q}(k, \gamma_{i+1}, \dots, \gamma_r)$ .  $\square$

**Lemma 5.2.** *We have the following:*

$$\tilde{q}(k, \gamma_1, \dots, \gamma_r) = \sum_{i=1}^r \llbracket k - \gamma_i + 1 \rrbracket - \sum_{i=1}^{r-1} \llbracket k - \gamma_{i+1} - \gamma_i + 1 \rrbracket.$$

*Proof.* We can do induction on  $r \geq 1$ . If  $r = 1$  we use i) of Lemma 5.1. Let us assume that the formula is true for  $1, 2, \dots, r-1$  and let us prove it for  $r$ .

By ii) of Lemma 5.1 we get:

$$\tilde{q}(k, \gamma_1, \dots, \gamma_r) = \tilde{q}(k, \gamma_1, \dots, \gamma_{r-1}) + \tilde{q}(k, \gamma_r) - \tilde{q}(k, \gamma_{r-1} + \gamma_r).$$

By i) of Lemma 5.1 we get:

$$\tilde{q}(k, \gamma_1, \dots, \gamma_r) = \tilde{q}(k, \gamma_1, \dots, \gamma_{r-1}) + \llbracket k - \gamma_r + 1 \rrbracket - \llbracket k - \gamma_{r-1} - \gamma_r + 1 \rrbracket.$$

By induction:

$$\tilde{q}(k, \gamma_1, \dots, \gamma_r) =$$

$$= \sum_{i=1}^{r-1} [[k - \gamma_i + 1]] - \sum_{i=1}^{r-2} [[k - \gamma_{i+1} - \gamma_i + 1]] + [[k - \gamma_r + 1]] - [[k - \gamma_{r-1} - \gamma_r + 1]],$$

i.e. our Lemma.  $\square$

Now we can prove the following:

**Proposition 5.3.** *For any partition  $(\gamma_1, \gamma_2, \dots, \gamma_r)$  and for any positive integer  $k$  we have:*

- i)  $r = 1$ ;  $q(k, \gamma_1) = 1$  if  $k = \gamma_1$ ,  $q(k, \gamma_1) = 0$  otherwise.*
- ii)  $r \geq 2$ ;  $q(k, \gamma_1, \gamma_2, \dots, \gamma_r) = \tilde{q}(k, \gamma_1, \dots, \gamma_r) - [[k - \gamma_1]] - [[k - \gamma_r]].$*

*Proof.* If  $r = 1$  the proof of *i)* is immediate. Let us assume  $r \geq 2$ . If  $k < \gamma_i$  for any  $i = 1, \dots, r$  then  $q(k, \gamma_1, \gamma_2, \dots, \gamma_r) = \tilde{q}(k, \gamma_1, \dots, \gamma_r) = 0$  and *ii)* is proved. Then we can assume that there exists at least an index  $i$  such that  $k \geq \gamma_i$ .

If  $k < \gamma_1$  and  $k < \gamma_r$  it is easy to see that  $q(k, \gamma_1, \gamma_2, \dots, \gamma_r) = \tilde{q}(k, \gamma_1, \dots, \gamma_r)$  and *ii)* is proved.

If  $k \geq \gamma_1$  and  $k < \gamma_r$  we have that the covering set  $\Lambda_j(k)$  for  $(\gamma_1, \gamma_2, \dots, \gamma_r)$  having the lower  $j$  is  $[a_1 + \gamma_1 - k, a_1 + \gamma_1 - 1 = b_1]$  and the covering sets  $\Lambda_j(k)$  with  $a_1 + \gamma_1 - k - 1 \leq j \leq a_1 - 2$  (which are in number of  $k - \gamma_1$ ) are the only ones that we have to consider in calculating  $\tilde{q}(k, \gamma_1, \dots, \gamma_r)$  but we have not to consider in calculating  $q(k, \gamma_1, \gamma_2, \dots, \gamma_r)$ , and *ii)* is proved in this case too.

If  $k < \gamma_1$  and  $k \geq \gamma_r$  we can argue as in the previous case by considering the higher  $j$  for which  $\Lambda_j(k)$  is a covering set for  $(\gamma_1, \gamma_2, \dots, \gamma_r)$  and *ii)* holds.

If  $k \geq \gamma_1$  and  $k \geq \gamma_r$  we can compute  $q(k, \gamma_1, \gamma_2, \dots, \gamma_r) - \tilde{q}(k, \gamma_1, \dots, \gamma_r)$  by arguing as in the previous two cases, separately for the left and the right side of  $[a_1, b_r]$ . As  $r \geq 2$ , there is no interference among the two sides.

In conclusion *ii)* holds in any case.  $\square$

Now we can compute the functions  $\varphi(k)$  and  $\Delta^2 \varphi(k)$  for any special vector space  $T$ .

**Theorem 5.4.** *Let  $T \subset S^d U$  be a special vector space of dimension  $e + 1$  and of type  $T = \partial^{b_1}(m_1) \oplus \partial^{b_2}(m_2) \oplus \dots \oplus \partial^{b_r}(m_r)$ , where every polynomial  $m_i$  is a monomial. Then, for any  $k \in [2, \lambda]$  where  $\lambda$  is the height of  $T$ , we have:*

$$\begin{aligned} \varphi(k) &= \sum_{i=1}^r (b_i - k + 1) + \lambda - k + 1 + \sum_{i=1}^{r-1} [[k - b_{i+1} - b_i - 3]] + [[k - b_1 - 2]] + [[k - b_r - 2]] \\ \Delta^2 \varphi(k) &= \sum_{i=1}^{r-1} \{1, \text{ if } k = b_i + b_{i+1} + 2, 0 \text{ otherwise}\} + \{1, \text{ if } k = b_1 + 1\} + \{1, \text{ if } k = b_r + 1\}. \end{aligned}$$

*Proof.* By Corollary 2.6 we have that  $\varphi(k) = \dim \partial^{-k} T + \dim \partial^{-k+1} \partial T - \dim(Q_k)$ .

By Proposition 3.2 *iv)* we have that  $\dim(\partial^{-k} T) = \sum_{i=1}^r [[b_i - k + 1]]$ .

As  $T$  is special,  $\partial T = \partial^{\lambda-1}(h)$ , where  $h$  is a suitable monomial, apex of  $T$ , and  $\lambda = \sum_{i=1}^r (b_i + 2) = e + 1 + r$  is the height of  $T$ . Hence, by Lemma 3.1,  $\dim \partial^{-k+1} \partial T = \dim \partial^{\lambda-k}(h) = [[\lambda - k + 1]] = \lambda - k + 1$  as  $k \leq \lambda$ . Moreover, as  $T$  is special,  $r \geq 2$  and the associated partition of  $\lambda$  is  $(\gamma_1, \gamma_2, \dots, \gamma_r) = (b_1 + 2, b_2 + 2, \dots, b_r + 2)$ ; hence  $\dim(Q_k) = q(k) = \sum_{i=1}^r [[k - b_i - 1]] - \sum_{i=1}^{r-1} [[k - b_{i+1} - b_i - 3]] - [[k - b_1 - 2]] - [[k - b_r - 2]]$ , by Lemma 5.2 and Proposition 5.3.

Recalling that, for any  $z \in \mathbb{Z}$ , we have  $[[z]] - [[-z]] = z$  we get our first formula.

As far concerning the second formula we have that  $\Delta^2$  is a linear operator and  $\Delta^2\phi(k) = 0$  for any linear function  $\phi(k)$ . Moreover  $\Delta^2([[k+z]]) = 1$  if  $k = -z - 1$  and  $\Delta^2([[k+z]]) = 0$  otherwise, for any fixed integer  $z$ . Hence the second formula follows.  $\square$

**Remark 5.1.** *The second formula of Theorem 5.4 can be written in a more compact way by adding two others integers independent from  $T$  : i.e.  $b_0 = b_{r+1} = -1$ . In this case*

$$\Delta^2\varphi(k) = \sum_{i=0}^r \{1, \text{ if } k = b_i + b_{i+1} + 2, 0 \text{ otherwise}\}.$$

**Remark 5.2.** *In the assumptions of Theorem 5.4, let us suppose that  $\lambda = d - 2$ , so that there are no integers  $c_i$  such that  $c_i = 0$  by Proposition 3.3 iv), then  $d = \lambda + 2 = e + r + 3$  and  $s - 1 = d - e - 2 = r + 1$ . In this case we have  $c_i = b_i + b_{i+1} + 2$  for any  $i = 0, \dots, r$  and we can determine the splitting type of  $\mathcal{N}_f$ . If  $d = \lambda + 2 + x$ , with  $x \geq 1$ , we have to add  $x$  zero integers to the set of  $\{c_i\}$ . So that we can compute the splitting type of  $\mathcal{N}_f$  for any special vector space  $T$ .*

The above Remark 5.2 joint with Proposition 2.4 and Proposition 15 of [A-R2] allows to compute the splitting type of any monomial rational curve  $f$  according to our definition in § 2. It is natural to ask whether, in this way, we can give examples of every possible splitting type for  $\mathcal{N}_f$ . Unfortunately the answer is not. Let us give an example.

Let us choose  $d = 12$ ,  $e = 6$ ,  $s = 5$ , then  $\text{rank}(\mathcal{N}_f) = 4$  and the splitting type of  $\mathcal{N}_f$  is given by four integers  $c_i \geq 0$  such that  $c_1 + c_2 + c_3 + c_4 = 2(e + 1) = 14$  (see § 2). For instance a possibility is  $(6, 4, 2, 2)$ . However this type cannot be obtained by a monomial curve  $C$ . Let  $T \subset S^{12}U$  be the proper vector subspace giving  $C$ . By Proposition 2.4 and Proposition 15 of [A-R2] it is immediate to see that for any non special  $T$  the splitting type of  $\mathcal{N}_f$  cannot be  $(6, 4, 2, 2)$ . Moreover, if  $T$  is special then necessarily  $r = 3$  and, by the formula given by Remark 5.2, we have that the the splitting type of  $\mathcal{N}_f$ , for any special  $T$ , is given by four positive integers such that there exists at least an ordering  $(x, y, z, w)$  with  $x + z = y + w$ . This is not true for  $(6, 4, 2, 2)$ .

## REFERENCES

- [A-R1] A.Alzati-R.Re: "PGL(2) actions on Grassmannians and projective construction of rational curves with given restricted tangent bundle" J. Pure Appl. Algebra **219** (2015) pp.1320-1335.
- [A-R2] A.Alzati-R.Re: "Irreducible components of Hilbert schemes of rational curves with given normal bundle", arXiv:1502.02521 [math.AG].

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